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CONSTRUCTION THEOREMS FOR POLYTOPES

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ABSTRACT

Certain construction theorems are represented, which facilitate an inductive combinatorial construction of polytopes. That is, applying the constructions to a d-polytope with n vertices, given combinatorially, one gets many combinatorial d-polytopes — and polytopes only — with $n+1$ vertices. The constructions are strong enough to yield from the 4-simplex all the 1330 4-polytopes with up to 8 vertices.

1. Introduction

One of the main techniques for an inductive construction of polytopes is the so-called "beneath-beyond" technique, used since Euler's times, and formalized by Grünbaum in [9, section 5.2]. Grünbaum's formulation is slightly erroneous, and therefore we reformulate it in Section 2.

Basically, if $Q \subset \mathbf{R}^d$ is a d-polytope, x is a point in $\mathbf{R}^d$ which lies outside Q, and P is the polytope $\operatorname{conv}(Q \cup \{x\})$, Grünbaum's theorem determines the facial structure of P from that of Q. In particular, if $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ is a partition of the facets of Q such that x lies in the affine hull of every $A \in \mathcal{A}$, beyond every $B \in \mathcal{B}$, and beneath every $C \in \mathcal{C}$, with respect to Q, then the combinatorial structure of P is determined merely by the triple $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and is independent of the exact location of the point x, provided that there exists at least one suitable point x.

The main obstacle in the use of Grünbaum's theorem is that for a given partition $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of the facets of Q, it is often very difficult to determine whether or not there exists such a suitable x. In fact, for the combinatorial study of polytopes it is usually sufficient to know that there is a polytope $Q' = \varphi(Q)$, where φ is a projective transformation permissible for Q, or, more generally, a combinatorial equivalence, such that there is a suitable point x for the triple $\varphi(\mathcal{A})$, $\varphi(\mathcal{B})$, $\varphi(\mathcal{C})$, with respect to Q'.

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Thus there arises the question: For a given d-polytope Q and a partition $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ of the facets of Q, what are the conditions on the combinatorial structure of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ that will guarantee the existence of a polytope $Q' = \varphi(Q)$, where φ is a combinatorial equivalence, and the existence of a point x, such that x lies in the affine hull of every $A \in \varphi(\mathscr{A})$, beyond every $B \in \varphi(\mathscr{B})$ and beneath every $C \in \varphi(\mathscr{C})$, with respect to $\varphi(Q)$?

Two of the few (and simple) cases for which the answer is known to be positive, are the cases in which $\mathscr{A}$ is empty and $\mathscr{B}$ consists of either a single facet of Q or of all the facets of Q that are in the star of F in the boundary complex of Q, where F is any proper face of Q (see e.g., [10, p. 445]). The full answer to our question is believed to be difficult, as it clearly leads to a solution for the famous "Steinitz problem". The Steinitz problem asks for conditions which are necessary and sufficient for a combinatorial $(d-1)$-sphere S (in the sense of [3]), $d \geqq 4$, in order that S be realizable as the boundary complex of a d-polytope, and is perhaps the main open problem in the combinatorial theory of polytopes.

In this paper we give a partial solution to the above question, in the form of three theorems (Theorems 1–3 in Section 4). Those theorems contain as particular cases most (if not all) of the partial solutions to our question which previously appeared in the literature. In the 4-dimensional case, they led to the construction of all the 4-polytopes with eight vertices (see [4]). Those three theorems were discovered by the first author while working on [3] and [4], and he also gave a direct proof for them. The second author suggested another approach, based on the sequence of lemmas given in Section 3. We preferred this approach, since it seems to us that those lemmas give a deeper background, from which perhaps more results can be drawn. In particular, see Remark 4 in Section 5.

In Section 2 we reformulate Grünbaum's theorem, and we introduce the basic concepts studied in the present paper. In Section 3 we present a sequence of lemmas which lead to the proofs of Theorems 1–3 in Section 4. Though here they serve mainly as a tool for proving Theorems 1–3, we believe that some of them may lead in the future to more results in the spirit of Theorems 1–3. We conclude in Section 5 with some remarks. In particular, in Remark 5 we discuss a relationship between the main concept in the present paper and the concept of shellability. Our notation follows [9].

2. Basic concepts

Let $Q \subset \mathbf{R}^d$ be a d-polytope, let x be a point in $\mathbf{R}^d$ outside Q, and let P be the polytope $\mathrm{conv}(Q \cup \{x\})$. The point x determines a partition $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ of the facets of Q such that x lie in the affine hull of every $A \in \mathcal{A}$, beyond every $B \in \mathcal{B}$ and beneath every $C \in \mathcal{C}$, with respect to Q. The relation between the facial structure of P and the triple $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ is determined by theorem 1 in [9, section 5.2]. However, as pointed out by M. A. Perles (private communication), that theorem is slightly erroneous in the sense that part (ii) in Grünbaum's formulation is incorrect. The same error appears also in the formulation of this theorem in [13, section 2.5]. We therefore prefer the following formulation.

THEOREM (Grünbaum). *Let $Q \subset \mathbf{R}^d$ be a d-polytope and let $x \in \mathbf{R}^d$ be a point outside Q. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be the partition of the facets of Q such that x lies in the affine hull of every $A \in \mathcal{A}$, beyond every $B \in \mathcal{B}$ and beneath every $C \in \mathcal{C}$. Define three types of sets G:*

(i) *G is a face of a member of $\mathcal{C}$.*

(ii) $G = \operatorname{conv}(F \cup \{x\})$, *where F is the intersection of a subset of $\mathcal{A}$ (or, equivalently, F is a face of Q and $x \in \operatorname{aff} F$).* ($\bigcap \varnothing = Q$.)

(iii) $G = \operatorname{conv}(F \cup \{x\})$, *where F is a face of a member of $\mathcal{B}$ and also a face of a member of $\mathcal{C}$.*

Then the set of types (i), (ii), (iii) *are faces of* $P = \operatorname{conv}(Q \cup \{x\})$, *and each face of P is of exactly one of the above types.*

As useful consequences of this Theorem, we mention:

(1) The facial structure of P depends only on the partition $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of the facets of Q induced by x, and not on the exact location of the point x.

(2) $\operatorname{vert} P = \operatorname{vert} Q \cup \{x\}$ if and only if $\mathcal{B} \neq \varnothing$ and $\operatorname{vert} Q \subset \bigcup \mathcal{C}$ (every vertex of Q has a facet in $\mathcal{C}$ that includes it).

In this case, if F is a face of Q, then

(3) F is a facet of P if and only if $F \in \mathcal{C}$.

(4) $\operatorname{conv}(F \cup \{x\})$ is a facet of P if and only if either $F \in \mathcal{A}$ or Q has exactly two facets which include F, one in $\mathcal{B}$ and the other in $\mathcal{C}$.

Consequence (1) motivates the following definitions.

DEFINITION. Let Q be a d-polytope in $\mathbf{R}^d$. Let $\mathcal{A}$, $\mathcal{B}$ be two disjoint collections of facets of Q. We say that a point x (in $\mathbf{R}^d$) *covers* the pair $\mathcal{B} \mid \mathcal{A}$ in Q if x lies beyond (with respect to Q) all the members of $\mathcal{B}$, $x \in \bigcap_{F \in \mathcal{A}} \operatorname{aff} F$ and x lies beneath all the other facets of Q. The pair $\mathcal{B} \mid \mathcal{A}$ is *coverable* in Q if there is a point x which covers $\mathcal{B} \mid \mathcal{A}$ in Q.

Since we are interested mainly in the combinatorial structure of Q and of the polytope obtained from Q by covering some pair $\mathcal{B} \mid \mathcal{A}$, it is natural to define:

DEFINITION. $\mathscr{B} \mid \mathscr{A}$ is C-*coverable* if there exists a combinatorial equivalence φ such that the induced pair $\varphi(\mathscr{B}) \mid \varphi(\mathscr{A})$ is coverable in $\varphi(Q)$. $\mathscr{B} \mid \mathscr{A}$ is P-*coverable* in Q if there exists a nonsingular projective transformation φ, permissible for Q, such that $\varphi(\mathscr{B}) \mid \varphi(\mathscr{A})$ is coverable in $\varphi(Q)$. (Clearly, a P-coverable pair is also C-coverable.)

The pair $\mathscr{B} \mid \mathscr{A}$ is called *strongly coverable* if $\varphi(\mathscr{B}) \mid \varphi(\mathscr{A})$ is coverable in $\varphi(Q)$ for every combinatorial equivalence φ.

Let Q be a d-polytope such that $0 \in \operatorname{int} Q$. The polar polytope $Q^* = \{y \in \mathbf{R}^d : \langle y, x \rangle \leqq 1 \text{ for all } x \in Q\}$ of Q is dual to Q by the inclusion reversing mapping $F \to \hat{F} = \{y \in Q^* : \langle y, x \rangle = 1 \text{ for all } x \in F\}$ between $\mathscr{F}(Q)$ and $\mathscr{F}(Q^*)$. For detailed discussion of polarity see [9, section 3.4] or [13, section 2.2].

For $\mathscr{D} \subset \mathscr{F}(Q)$, define: $\hat{\mathscr{D}} = \{\hat{F} : F \in \mathscr{D}\}$.

Let J be a nonempty proper face of Q. Then there is a $(d - 1 - \dim J)$-flat H such that $\bar{Q} = Q \cap H$ is a $(d - 1 - \dim J)$-polytope and the mapping $F \to \bar{F} = F \cap H$ is an isomorphism between the segment $[J, Q]$ of the lattice $\mathscr{F}(Q)$ and the lattice $\mathscr{F}(Q \cap H)$. The polytope $\bar{Q}$ is called the *quotient polytope of* Q with respect to J, and also the *face figure* of Q at J (*vertex-figure* if J is a vertex of Q) and is denoted also by Q/J. (See [1, p. 98], [13, p. 71] and [9, exercise 3.4.10(iii)].) Note that $J \cap H = \varnothing$.

3. Properties of coverable pairs

The present section contains a sequence of Lemmas. In Lemmas 1–3 we investigate the connection between the coverability of $\mathscr{B} \mid \mathscr{A}$ in Q and the existence of a certain hyperplane intersecting the polar polytope Q^*. This facilitates the proof of Theorem 1 in Section 4.

In Lemmas 4–6 we study the case in which all the members of $\mathscr{A} \cup \mathscr{B}$ share a common vertex. We study the relationship between coverability in Q and P-coverability in a face figure of Q. This enables us to start from P-coverability in a low dimensional polytope, and to conclude about coverability in a higher dimensional polytope.

Throughout the sequel Q is a d-polytope, $0 \in \operatorname{int} Q$ and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ is a partition of the set of all facets of Q.

Let x be a point in $\mathbf{R}^d$. Consider the hyperplane $H = \{y \in \mathbf{R}^d : \langle y, x \rangle = 1\}$. H determines two open half-spaces $H^+ = \{y \in \mathbf{R}^d : \langle y, x \rangle > 1\}$, $H^- = \{y \in \mathbf{R}^d : \langle y, x \rangle < 1\}$. Clearly, x covers $\mathscr{B} \mid \mathscr{A}$ if and only if $\hat{\mathscr{A}} \subset H$, $\hat{\mathscr{B}} \subset H^+$, $0 \in H^-$ and $\hat{\mathscr{C}} \subset H^-$. This proves:

LEMMA 1. *The following statements are equivalent*:

(a) $\mathscr{B} \mid \mathscr{A}$ *is coverable in* Q.

(b) *There is a hyperplane which includes* $\hat{\mathscr{A}}$ *and strictly separates* $\hat{\mathscr{B}}$ *and* $\hat{\mathscr{C}} \cup \{0\}$. ∎

The following lemma is an immediate consequence of Lemma 1.

LEMMA 2. *The following statements are equivalent*:

(a) *Either* $\mathscr{B} \mid \mathscr{A}$ *or* $\mathscr{C} \mid \mathscr{A}$ *is coverable in* Q.

(b) $\hat{\mathscr{B}}$ *and* $\hat{\mathscr{C}}$ *can be strictly separated by a hyperplane which includes* $\hat{\mathscr{A}}$ *and does not pass through* 0. ∎

For a point c in $\operatorname{int} Q^*$, define a transformation T_c by:

$$T_c x = \frac{x}{1-\langle c, x\rangle}.$$

T_c is a nonsingular projective transformation, permissible for Q.

Let T be a nonsingular projective transformation permissible for Q. Then T has a presentation

$$Tx = \frac{Mx + b}{\delta - \langle c, x\rangle}$$

where M is a regular $d \times d$ matrix, b and c are points in $\mathbf{R}^d$ and $\delta \in \mathbf{R}$. $\delta \neq 0$ since $T0$ is defined. Assume, w.l.o.g., that $\delta = 1$. $1 - \langle c, 0\rangle = 1$, hence, for every x in Q, $1 - \langle c, x\rangle > 0$. Thus $c \in \operatorname{int} Q^*$. Define a transformation S by

$$Sx = Mx + (1 + \langle c, x\rangle)b.$$

S is a regular affine transformation and

$$ST_c = S\frac{x}{1-\langle c, x\rangle} = \frac{Mx}{1-\langle c, x\rangle} + b + \frac{\langle c, x\rangle b}{1-\langle c, x\rangle} = Tx.$$

It is easy to see that if S is a regular affine transformation, then $\mathscr{B} \mid \mathscr{A}$ is coverable in Q if and only if $S\mathscr{B} \mid S\mathscr{A}$ is coverable in SQ. It follows that $\mathscr{B} \mid \mathscr{A}$ is P-coverable in Q if and only if $T_c\mathscr{B} \mid T_c\mathscr{A}$ is coverable in T_cQ for some $c \in \operatorname{int} Q^*$, where

$$T_c x = \frac{x}{1-\langle c, x\rangle}.$$

LEMMA 3. *Assume that* $\mathscr{B}$ *and* $\mathscr{C}$ *are not empty. The following statements are equivalent*:

(a) $\mathscr{B} \mid \mathscr{A}$ *is* P-*coverable in* Q.
(b) $\mathscr{C} \mid \mathscr{A}$ *is* P-*coverable in* Q.
(c) $\hat{\mathscr{B}}$ *and* $\hat{\mathscr{C}}$ *can be strictly separated by a hyperplane which includes* $\hat{\mathscr{A}}$.

PROOF. Assume that (a) holds. Then $T_c\mathscr{B} \mid T_c\mathscr{A}$ is coverable in T_cQ for some $c \in \operatorname{int} Q^*$. Note that $(T_cQ)^* = Q^* - c$ (see [9, exercise 3.4.4]). It is easy to check that if F is a facet of Q, then the vertex of $Q^* - c$ which corresponds to the facet T_cF of T_cQ is $\hat{F} - c$.

Hence, by Lemma 1 (applied to T_cQ), there exists a hyperplane H which includes $\{\hat{F} - c : F \in \mathscr{A}\}$ and strictly separates $\{\hat{F} - c : F \in \mathscr{B}\}$ and $\{\hat{F} - c : F \in \mathscr{C}\}$. Therefore (c) holds.

Assume that (c) holds. Let H be a hyperplane which includes $\hat{\mathscr{A}}$ and strictly separates $\hat{\mathscr{B}}$ and $\hat{\mathscr{C}}$. Choose a point $c \in \operatorname{int} Q^*$ such that $H - c$ strictly separates $\hat{\mathscr{B}} - c$ and $(\hat{\mathscr{C}} - c) \cup \{0\}$ (note that $\mathscr{C} \neq \emptyset$). By Lemma 1, $T_c\mathscr{B} \mid T_c\mathscr{A}$ is coverable in $(Q^* - c)^*$. ∎

The following Lemmas will enable us to add a vertex to Q by adding a vertex to a face figure of Q, which is a polytope of a lower dimension.

LEMMA 4. *Assume that* $\mathscr{B} \mid \mathscr{A}$ *is* P-*coverable in* Q. *If there is a common vertex to all members of* $\mathscr{A} \cup \mathscr{B}$, *then* $\mathscr{B} \mid \mathscr{A}$ *is coverable in* Q.

PROOF. Assume that $p \in \operatorname{vert} F$ for every member F of $\mathscr{A} \cup \mathscr{B}$, $c \in \operatorname{int} Q^*$ and y covers $T_c\mathscr{B} \mid T_c\mathscr{A}$ in T_cQ. Denote by H^+ the open halfspace $\{x : 1 - \langle c, x\rangle > 0\}$. For $\varepsilon > 0$, define: $z = \varepsilon y + (1 - \varepsilon)T_cp$. Obviously, z covers $T_c\mathscr{B} \mid T_c\mathscr{A}$ in T_cQ. If ε is sufficiently small, then there is a point x in H^+ such that $z = T_cx$ (in fact $x = z/(1 + \langle c, z\rangle)$), and for every facet F of Q that includes p, the line $\operatorname{aff}\{0, x\}$ intersects $\operatorname{aff} F$ in H^+. T_c is permissible for H^+, thus T_c maps every open segment (k, k') in H^+ onto (T_ck, T_ck').

We shall prove that for $\varepsilon > 0$ sufficiently small, the point x covers $\mathscr{B} \mid \mathscr{A}$. Let F be a facet of Q not in $\mathscr{A} \cup \mathscr{B}$, such that $p \in F$. We have to show that x lies beneath F with respect to Q. z lies beneath T_cF with respect to T_cQ. Thus there is a point $z' \in \operatorname{aff} T_cF$ such that z is interior to the line segment $[0, z']$. $T_c^{-1}(z')$ is in $\operatorname{aff} F$, and, as z' is close to T_cp, it is close to p and therefore lies in H^+. Since T_c is permissible for H^+, $x = T_c^{-1}(z)$ is interior to the line segment $[0, T_c^{-1}(z')]$, and lies therefore beneath F with respect to Q. In a similar manner one can see that x lies beyond every $B \in \mathscr{B}$ and in the affine hull of every $A \in \mathscr{A}$.

If ε is sufficiently small, then x is sufficiently close to p, and therefore x lies beneath all the facets of Q which do not include p. ∎

DEFINITION. Let R be a subset of Q. Define:

$$\mathcal{F}_R = \mathcal{F}_R(Q) = \{F : F \text{ is a facet of } Q, F \supset R\}.$$

Let J be a nonempty proper face of Q, and let H be a suitable flat such that $\bar{Q} = Q \cap H$ is the face figure Q/J. For a face F in $\mathcal{F}_J$, let $\bar{F}$ denote the face $F \cap H$ of $\bar{Q}$.

Assume in the sequel that $\mathcal{A}$ and $\mathcal{B}$ are disjoint subsets of $\mathcal{F}_J$.

LEMMA 5. (a) *The pair* $\emptyset \mid \mathcal{A}$ *is coverable in* Q.

(b) *If the pair* $\mathcal{B} \mid \mathcal{A}$ *is coverable in* Q, *then the pair* $\mathcal{F}_J \setminus (\mathcal{B} \cup \mathcal{A}) \mid \mathcal{A}$ *too is coverable in* Q.

PROOF. Choose points $p \in \operatorname{relint} J$, $a \in \operatorname{relint} \cap \mathcal{A}$ (we adopt the convention $\bigcap \emptyset = Q$) and a point x that covers $\mathcal{B} \mid \mathcal{A}$. The point a covers $\emptyset \mid \mathcal{A}$. For $\varepsilon > 0$ sufficiently small, the point $(1+\varepsilon)p - \varepsilon x$ covers $\mathcal{F}_J \setminus (\mathcal{B} \cup \mathcal{A}) \mid \mathcal{A}$. ■

REMARK. Assume that $F \in \mathcal{F}_J$. It is clear that a point x in H lies beyond F with respect to Q if and only if x lies beyond $\bar{F}$ with respect to $\bar{Q}$.

LEMMA 6. *If* $\{\bar{F} : F \in \mathcal{B}\} \mid \{\bar{F} : F \in \mathcal{A}\}$ *is* P-*coverable in* $\bar{Q}$, *then* $\mathcal{B} \mid \mathcal{A}$ *and* $\mathcal{F}_J \setminus (\mathcal{B} \cup \mathcal{A}) \mid \mathcal{A}$ *are coverable in* Q.

PROOF. Assume: $\bar{T} : H \to H$ is a regular projective transformation, permissible for $\bar{Q}$, and $x \in H$ covers $\{\bar{T}\bar{F} : F \in \mathcal{B}\} \mid \{\bar{T}\bar{F} : F \in \mathcal{A}\}$ in $\bar{T}\bar{Q}$. There is a regular projective transformation T, permissible for Q, such that $T|_H = \bar{T}$. It is easy to check that if $p \in \operatorname{relint} TJ$ and $\varepsilon > 0$ is sufficiently small, then $\varepsilon x + (1-\varepsilon)p$ covers $T\mathcal{B} \mid T\mathcal{A}$ in TQ. By Lemma 4, $\mathcal{B} \mid \mathcal{A}$ is coverable in Q.

By Lemma 5 the pair $\mathcal{F}_J \setminus (\mathcal{B} \cup \mathcal{A}) \mid \mathcal{A}$ is coverable too. ■

In the sequel we shall need the following trivial lemma.

LEMMA 7. *Let* P *be an* m-*gon* (*in* $\mathbf{R}^2$). *Let* $E_1, \cdots, E_m$ *be all its edges in their natural cyclic order. Then, up to authomorphisms of* P, *exactly the following pairs are* P-*coverable in* P:

(1) $\{E_1, \cdots, E_n\} \mid \emptyset$, $0 \leqq n < m$,

(2) $\{E_2, \cdots, E_n\} \mid \{E_1\}$, $1 \leqq n < m$,

(3) $\{E_2, \cdots, E_{n-1}\} \mid \{E_i, E_n\}$, $2 \leqq n < m$. ■

4. Construction theorems

The previous lemmas lead to the following three theorems.

THEOREM 1. *Let Q be a d-polytope, $d \geqq 2$. Let F be a facet of Q which is a simplex. Denote by $\mathscr{D}$ the set of facets of Q adjacent to F ($|\mathscr{D}| = d$). Assume that $\mathscr{R}$ and $\mathscr{S}$ are disjoint subsets of $\mathscr{D}$ (possibly empty). If Q is not a simplex or if $\mathscr{R} \cup \mathscr{S} \neq \mathscr{D}$, then $\{F\} \cup \mathscr{S} \mid \mathscr{R}$ is* P-*coverable in Q. Moreover, if $|\mathscr{R} \cup \mathscr{S}| \leqq d-2$, or none of the vertices of F is simple in Q (that is, of valence d), and if x covers $T\mathscr{B} \mid T\mathscr{A}$ in TQ (where T is a regular projective transformation permissible for Q), then* $\operatorname{vert}(\operatorname{conv}(TQ \cup \{x\})) = \operatorname{vert} TQ \cup \{x\}$.

PROOF. Consider the polytope Q^* ($0 \in \operatorname{int} Q$). Denote by H the hyperplane spanned by the vertices $\hat{G}$ of Q^*, $G \in \mathscr{D}$. Assume that $\hat{F} \in H^+$. Clearly $\hat{F}$ is the only vertex of Q^* lying in H^+. Applying a small perturbation to H, it passes through $\{\hat{G} : G \in \mathscr{R}\}$ and $\operatorname{vert} Q^* \cap H^+ = \{\hat{F}\} \cup \hat{\mathscr{S}}$. The assertions now follow from Lemma 3 and consequence (2) of Grünbaum's theorem (in Section 1). ■

A particular case of Theorem 1 appeared in [14] and in [12].

THEOREM 2. *Let $Q \subset \mathbf{R}^d$ ($d \geqq 4$) be a d-polytope, let L be a $(d-3)$-face of Q and let $F_1, \cdots, F_m$ be all the facets of Q in* $\operatorname{st}(L, Q)$ *in their natural cyclic order, that is, $F_i \cap F_{i+1}$ ($1 \leqq i \leqq m$, where $F_{m+1} = F_1$) is the $(d-2)$-face common to F_i, F_{i+1} and contains L. Let p be a vertex of L and let $\mathscr{D}$ be the set of all the facets of Q which contain p. Then the following pairs are strongly coverable in Q:*

(1) $\{F_1, \cdots, F_n\} \mid \varnothing$ $(1 \leqq n \leqq m)$.
(2) $\{F_2, \cdots, F_n\} \mid \{F_1\}$ $(2 \leqq n \leqq m)$.
(3) $\{F_2, \cdots, F_{n-1}\} \mid \{F_1, F_n\}$ $(3 \leqq n \leqq m)$.
(4) $\mathscr{D} \setminus \{F_1, \cdots, F_n\} \mid \varnothing$ $(1 \leqq n \leqq m)$.
(5) $\mathscr{D} \setminus \{F_1, \cdots, F_n\} \mid \{F_1\}$ $(2 \leqq n \leqq m)$.
(6) $\mathscr{D} \setminus \{F_1, \cdots, F_n\} \mid \{F_1, F_n\}$ $(3 \leqq n \leqq m)$.

In every case, if x covers the suitable pair $\mathscr{B} \mid \mathscr{A}$, then $\operatorname{vert}(\operatorname{conv}(Q \cup \{x\})) = \operatorname{vert} Q \cup \{x\}$.

PROOF. (1)–(3), $n < m$. Apply Lemma 7 to a face figure of Q at L, and Lemma 6 with $J = L$.

(1)–(3), $n = m$. Apply both parts of Lemma 5 with $J = L$.

(4)–(6). These cases follow from the cases (1)–(3) by Lemma 5 with $J = \{p\}$.

So far we proved coverability. However, since our proof used just the combinatorial structure — rather than the geometric structure — of $\mathscr{F}(Q)$, the strong coverability follows.

Suppose x covers a pair $\mathscr{B} \mid \mathscr{A}$ of one of the six cases. $\mathscr{B} \neq \varnothing$, thus $x \notin Q$. In all the six cases, Q has a facet F that contains p and does not belong to $\mathscr{B} \cup \mathscr{A}$. If q is a vertex of Q, other than p, then Q has a facet F' such that $q \in F'$ and $p \notin F'$.

Obviously $F' \not\in \mathcal{B} \cup \mathcal{A}$. By consequence (2) of Grünbaum's theorem, $\text{vert}(\text{conv}(Q \cup \{x\})) = \text{vert}\, Q \cup \{x\}$. ∎

A particular case of Theorem 2(1) with $d = 4$ was already known to Grünbaum and Sreedharan [10, p. 445].

THEOREM 3. *Let $Q \subset \mathbf{R}^d$ be a d-polytope, $d \geqq 3$, let F be a facet of Q and let p be a vertex of F. Define*: $\mathcal{E} = \{G : G$ *is a facet of* Q, $p \in G$, $G \neq F\}$, $\mathcal{D} = \{G \in \mathcal{E} : \dim(G \cap F) = d - 2\}$. *($\mathcal{D}$ is just the set of facets of Q adjacent to F and containing p.) Assume*: $|\mathcal{D}| = d - 1$. *(That is, the valence of p in F is $d - 1$.)*

Then for every two disjoint subsets $\mathcal{S}$, $\mathcal{R}$ of $\mathcal{D}$ with $\mathcal{S} \cup \mathcal{R} \neq \mathcal{E}$ (including the cases $\mathcal{S} = \emptyset$, $\mathcal{R} = \emptyset$) the following pairs are strongly coverable in Q:

(1) $\{F\} \cup \mathcal{S} \,|\, \mathcal{R}$,

(2) $\mathcal{E} \setminus (\mathcal{S} \cup \mathcal{R}) \,|\, \mathcal{R}$.

In both cases, if x covers the suitable pair $\mathcal{B} \,|\, \mathcal{A}$, then $\text{vert}(\text{conv}(Q \cup \{x\})) = \text{vert}\, Q \cup \{x\}$.

PROOF. We shall prove coverability:

(1) $\mathcal{R} \cup \mathcal{S} \neq \mathcal{E}$, hence either $\mathcal{R} \cup \mathcal{S} \neq \mathcal{D}$ or $\mathcal{D} \neq \mathcal{E}$ (the vertex p is not simple). Apply Theorem 1 to a vertex figure of Q at p, and use Lemma 6.

(2) Follows from (1) by Lemma 5(b) with $J = \{p\}$.

The strong coverability and the assertion about the vertices of $\text{conv}(Q \cup \{x\})$ follow in a manner similar to the proper part of the proof of Theorem 2. ∎

5. Remarks

(1) Theorems 1, 2, 3 have been programmed and used in [4] for an inductive construction of 4-polytopes with 8 vertices from those with 7 vertices. Surprisingly, they yielded all the 1294 4-polytopes with 8 vertices. Moreover, the programmed version of Theorems 1, 2, 3 has been repeatedly applied to the 4-simplex, and yielded the 4 4-polytopes with 6 vertices, and the 31 4-polytopes with 7 vertices.

(2) If Q is a rational polytope (see [9, p. 92]) and Q' is the polytope $\text{conv}(Q \cup \{x\})$ obtained from Q by means of any of our Theorems 1, 2, 3, then Q' too is a rational polytope. This follows from the fact, which is easily verifiable, that each of the Lemmas 1–7 is correct also if we replace the Euclidean space $\mathbf{R}^d$ by the rational space $\mathbf{Q}^d$.

The fact that there exists a non-rational polytope (the "smallest" known is of dimension 8 and had 12 vertices [9, p. 94]), shows that not all the d-polytopes can be obtained from a d-simplex by a repeated application of our Theorems and

Lemmas. In fact, one can easily check that, following Altshuler's notation in [2], the polytopes N^9_{20} and N^9_{22} cannot be constructed by our Theorems or Lemmas.

(3) Note that in parts (1), (2) and (3) of Theorem 2, the strong coverability of each of the discussed pairs is completely independent of the combinatorial structure of the facets of Q that are not in the pair. This motivates the following definitions:

Let Q be a d-polytope and let $\mathcal{B} \mid \mathcal{A}$ be a coverable pair in Q. We say that the pair $\mathcal{B} \mid \mathcal{A}$ is *universally coverable* if for every d-polytope Q' such that there is an injection φ from $\mathcal{A} \cup \mathcal{B}$ into the set of facets of Q' with the property that $\dim \bigcap \varphi(\mathcal{C}) = \dim \bigcap \mathcal{C}$ for every $\mathcal{C} \subset \mathcal{A} \cup \mathcal{B}$, the pair $\varphi(\mathcal{B}) \mid \varphi(\mathcal{A})$ is coverable in Q'. The concepts *universally* P-*coverable* pair and *universally* C-*coverable* pair are defined similarly.

Obviously, universal coverability implies strong coverability.

It is easy to see that each of the pairs in parts (1), (2) and (3) of Theorem 2 is universally coverable; in the notation of Theorem 3, the pair $\{F\} \cup \mathcal{S} \mid \mathcal{D} \setminus \mathcal{S}$ is universally coverable and in the notation of Theorem 1, if Q is not a simplex then the pair $\{F\} \cup \mathcal{S} \mid \mathcal{D} \setminus \mathcal{S}$ is universally P-coverable.

(4) Lemma 5 can be generalized as follows:

Let $J_1 \subset \cdots \subset J_k = J$ $(k \geqq 1)$ be a strictly increasing sequence of non-empty proper faces of Q. Assume that $\mathcal{A}$ and $\mathcal{B}$ are two disjoint subsets of $\mathcal{F}_J$. Define:

$$\mathcal{F}_j = \mathcal{F}_{J_j},$$

$$\mathcal{C} = \mathcal{F}_1 \setminus (\mathcal{F}_2 \setminus (\cdots \setminus (\mathcal{F}_k \setminus (\mathcal{B} \cup \mathcal{A})) \cdots)).$$

LEMMA 8. *If* $\{\bar{F} : F \in \mathcal{B}\} \mid \{\bar{F} : F \in \mathcal{A}\}$ *is* P-*coverable in* $\bar{Q}$ *then* $\mathcal{C} \mid \mathcal{A}$ *is coverable in* Q.

PROOF. By induction on k, using the same method of Lemma 5. ∎

The construction of Lemma 8 is called *sewing* through the *k-tower* $J_1, \cdots, J_k$. It enables one to obtain additional theorems, in dimension d, from an existing theorem in a lower dimension. Our Theorems can be viewed as particular cases of sewing with $k = 1$ and $k = 2$. For example, Theorem 2(4) with $n = m$ is sewing through the 2-tower $J_1 = \{p\}$, $J_2 = L$, with $\mathcal{B} = \mathcal{A} = \emptyset$.

In [14] the second author developed the sewing technique for constructing neighborly $2m$-polytopes $(m \geqq 2)$. The "facet-splitting" operation of Barnette [6] is dual to the sewing construction. The relation between these two constructions is discussed in [14, section 7.4].

(5) In Lemmas 1–3 we gave some necessary properties of coverability. Another necessary property of a coverable pair is its shellability. (Consult [7] and [8] for definitions.)

LEMMA 9. *Assume $\mathcal{B} \mid \mathcal{A}$ is C-coverable in Q. Then Q has a shelling*

$$F_1, \cdots, F_r, \cdots, F_s, \cdots, F_t, \qquad 0 \leqq r \leqq s < t$$

such that $\mathcal{B} = \{F_1, \cdots, F_r\}$, $\mathcal{A} = \{F_{r+1}, \cdots, F_s\}$ and $\{F_1, \cdots, F_t\}$ is the set of all facets of Q.

PROOF. We give the outline of the proof, based on lemma 2.1 in [7]. W.l.o.g., $\mathcal{B} \mid \mathcal{A}$ is coverable in Q. By Lemma 1 there is a point h such that $\langle h, \hat{F} \rangle < 1$ for $F \in \mathcal{B}$, $\langle h, \hat{F} \rangle = 1$ for $F \in \mathcal{A}$ and $\langle h, \hat{F} \rangle > 1$ otherwise. Choose a point k such that $\langle k, \hat{F} \rangle \neq \langle k, \hat{F}' \rangle$ whenever $F \neq F'$. For $\lambda > 0$, define: $f = h + \lambda k$. Define also: $r = |\mathcal{B}|$, $s = r + |\mathcal{A}|$. If λ is sufficiently small, then there is an ordering $F_1, \cdots, F_t$ of all the facets of Q, such that $\langle f, \hat{F}_i \rangle < \langle f, \hat{F}_j \rangle$ for $1 \leqq i < j \leqq t$, $\mathcal{B} = \{F_1, \cdots, F_r\}$ and $\mathcal{A} = \{F_{r+1}, \cdots, F_s\}$. By lemma 2.1 in [7] $F_1, \cdots, F_t$ is a shelling. ■

(6) The problem of C-coverability for $d = 3$ is partially settled by Barnette's theorem [5]. Let P be a 3-polytope and let $\mathcal{B}$ be a non-empty set of facets of P. An edge E of P is called a boundary edge of $\mathcal{B}$ if E is an edge of exactly one member of $\mathcal{B}$. Then $\mathcal{B} \mid \varnothing$ is C-coverable in P if and only if the graph spanned by the boundary edges of $\mathcal{B}$ is a (non-empty) simple circuit.

(7) By a result due to Shephard (see [16] or [11]), we obtain a similar result for d-polytopes P with $d + 2$ vertices. Let $\mathcal{B}$ be a non-empty set of facets of P. Then $\mathcal{B} \mid \varnothing$ is P-coverable in P if and only if the set $\bigcup \mathcal{B}$ is a $(d-1)$-ball.

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